

VITERBO'S TRANSFER MORPHISM FOR SYMPLECTOMORPHISMS

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ABSTRACT. We construct an analogue of Viterbo's transfer morphism for Floer homology of an automorphism of a Liouville domain. As an application we prove that the Dehn-Seidel twist along *any* Lagrangian sphere in a Liouville domain of dimension ≥ 4 has infinite order in the symplectic mapping class group.

1. INTRODUCTION

In his 1999 paper [8], Viterbo constructed a morphism

$$(1.1) \quad SH_*(W_2) \rightarrow SH_*(W_1)$$

associated to a codimension-0 embedding $W_1 \rightarrow W_2$ of a Liouville domain into another Liouville domain that preserves the Liouville form. The map (1.1), called the transfer morphism, fits into the commutative diagram

$$\begin{array}{ccc} SH_*(W_2) & \longrightarrow & SH_*(W_1) \\ \uparrow & & \uparrow \\ H_{*+n}(W_2, \partial W_2) & \longrightarrow & H_{*+n}(W_1, \partial W_1). \end{array}$$

Here, $2n$ is the dimension of W_2 and the map

$$H_{*+n}(W_2, \partial W_2) \rightarrow H_{*+n}(W_1, \partial W_1)$$

is the composition of the homomorphism induced by the inclusion $(W_2, \partial W_2) \hookrightarrow (W_2, W_2 \setminus W_1)$ and the excision isomorphism

$$H_{*+n}(W_2, W_2 \setminus W_1) \rightarrow H_{*+n}(W_1, \partial W_1).$$

The aim of this paper is to construct an analogue of Viterbo's transfer morphism for the groups $HF_*(\phi, a)$ associated to an exact symplectomorphism ϕ of a Liouville domain and a slope a (see [7]). As an application we construct an asymptotic growth invariant for such symplectomorphisms. As a byproduct we prove that the Dehn-Seidel twist along *any* Lagrangian sphere in a Liouville domain of dimension ≥ 4 is of infinite order in the symplectic mapping class group (see Corollary 1.4 below).

Let $a \leq a'$ be two admissible slopes (i.e. a, a' are not periods of any Reeb orbit on the boundary of the Liouville domain). There is a well defined continuation map

$$HF_*(\phi, a) \rightarrow HF_*(\phi, a').$$

Given W_1 and W_2 as above, an exact symplectomorphism ϕ of W_1 can also be seen as an exact symplectomorphism of W_2 . To avoid any ambiguity, we will write $HF_*(W_j, \phi, a)$ for the group $HF_*(\phi, a)$ when ϕ is seen as a symplectomorphism of W_j , $j \in \{1, 2\}$.

Theorem 1.1. *Let W_1, W_2, ϕ be as above, and let $a, b \in \mathbb{R}^+ \cup \{\infty\}$ be positive admissible slopes (with respect to W_2 and W_1 , respectively). Assume $a \leq b$. Then, there exists a linear map*

$$HF_*(W_2, \phi, a) \rightarrow HF_*(W_1, \phi, b),$$

*called the **transfer morphism**, with the following properties. It coincides with the map (1.1) for $a = b = \infty$, and ϕ equal to the identity. Moreover, the diagram*

$$(1.2) \quad \begin{array}{ccc} HF_*(W_2, \phi, a) & \longrightarrow & HF_*(W_1, \phi, b) \\ \downarrow & & \downarrow \\ HF_*(W_2, \phi, a') & \longrightarrow & HF_*(W_1, \phi, b'), \end{array}$$

consisting of transfer morphisms and continuation maps, commutes for all admissible slopes $a', b' \in \mathbb{R}^+ \cup \{\infty\}$ such that $a \leq a'$, $b \leq b'$, and $a' \leq b'$.

1.1. Applications. As an application, we construct a numerical invariant

$$\kappa(W, \phi) := \limsup_{m \rightarrow \infty} \frac{\dim HF(W, \phi^m, \varepsilon)}{m}$$

for an exact symplectomorphism ϕ of a Liouville domain W , the so-called iterated ratio (see also Definition 4.1). It does not depend on the ambient Liouville domain in the following sense.

Theorem 1.2. *Let W_1 and W_2 be Liouville domains as in Theorem 1.1 and let $\phi : W_1 \rightarrow W_2$ be an exact symplectomorphism, then $\kappa(W_1, \phi) = \kappa(W_2, \phi)$.*

The next theorem calculates κ for the square of a Dehn-Seidel twist furnished by a Lagrangian sphere in a Liouville domain.

Theorem 1.3. *Let W be a Liouville domain of dimension $2n \geq 4$, and let $L \subset W$ be a Lagrangian sphere. Then,*

$$\kappa(W, \tau_L^2) = 4,$$

where $\tau_L : W \rightarrow W$ stands for a Dehn-Seidel twist furnished by L .

Note that $\kappa(W, \text{id}) = 0$.

Corollary 1.4. *Any Lagrangian sphere L in any Liouville domain W of dimension greater than or equal to 4 gives rise to a Dehn-Seidel twist τ_L that represents an element of infinite order in the symplectic mapping class group. In other words, τ_L^k is not symplectically isotopic to the identity relative to the boundary for all $k \in \mathbb{N}$.*

Remark 1.5. The analogous (to Corollary 1.4) statement for closed symplectic manifolds is false. For instance, the Dehn-Seidel twist furnished by the Lagrangian diagonal in $\mathbb{S}^2 \times \mathbb{S}^2$ is symplectically isotopic to the identity.

Theorem 1.1 (applied to ϕ equal to the identity) can be used to obtain some information about closed geodesics.

Corollary 1.6. *Let (\mathbb{S}^n, g_0) be the n -dimensional sphere with the standard Riemannian metric, and let g be a Riemannian metric on \mathbb{S}^n such that $g \leq g_0$. Then, there exists a non-constant closed geodesic on (\mathbb{S}^n, g) of length less than or equal to 2π .*

We will prove, in fact, a more general statement (see Theorem 4.7 and Corollary 4.8).

1.2. Conventions. Let (W, λ) be a symplectic manifold. A function $H : W \rightarrow \mathbb{R}$ and its Hamiltonian vector field X_H are related by $dH = \omega(X_H, \cdot)$. The Hamiltonian isotopy $\psi_t^H : W \rightarrow W$ of a time-dependent Hamiltonian $H : \mathbb{R} \times W \rightarrow \mathbb{R} : (t, x) \mapsto H_t(x)$ is determined by $\partial_t \psi_t^H = X_{H_t} \circ \psi_t^H$, $\psi_0^H = \text{id}$. An almost complex structures J on W is said to be ω -compatible if $\omega(\cdot, J\cdot)$ is a Riemannian metric.

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2. PRELIMINARIES

2.1. Liouville domains.

Definition 2.1. A Liouville domain is a compact manifold W with a 1-form λ , called Liouville form, that satisfies the following conditions. The 2-form $d\lambda$ is a symplectic form on W , and the Liouville vector field X_λ , defined by $X_\lambda \lrcorner d\lambda = \lambda$, points transversally out on the boundary ∂W .

If (W, λ) is a Liouville domain, then the restriction $\lambda|_{\partial W}$ of λ to the boundary ∂W is a contact form. Moreover, a collar neighbourhood of ∂W can be identified with $(\partial W \times (0, 1], r \lambda|_{\partial W})$ using the flow of the Liouville vector field X_λ . Here, r stands for the real coordinate.

Definition 2.2. Let (W, λ) be a Liouville domain. There exists a unique embedding

$$\iota : \partial W \times (0, 1] \rightarrow W$$

such that $\iota(x, 1) = x$ and $\iota^* \lambda = r \lambda|_{\partial W}$. The **completion** \widehat{W} of (W, λ) is the exact symplectic manifold obtained by gluing W and $\partial W \times (0, \infty)$ via ι .

If W is a Liouville domain and $r \in (0, \infty)$, we denote by W^r the subset of the completion \widehat{W} defined by

$$W^r := \begin{cases} W \setminus (\partial W \times (r, 1]) & \text{for } r < 1 \\ W \cup \partial W \times (1, r] & \text{for } r \geq 1 \end{cases}.$$

Here, and in the rest of the paper, the sets W and $\partial W \times (0, \infty)$ are identified with the corresponding regions in the completion \widehat{W} .

2.2. Floer homology for exact symplectomorphisms.

Definition 2.3. An **exact symplectomorphism** ϕ of a Liouville domain W with connected boundary is a diffeomorphism of W equal to the identity near the boundary such that the 1-form $\phi^* \lambda - \lambda$ is exact. We denote by $F_\phi : W \rightarrow \mathbb{R}$ the unique function that is equal to 0 on the boundary and satisfy $dF_\phi = \phi^* \lambda - \lambda$.

Definition 2.4. Let ϕ be an exact symplectomorphism of a Liouville domain (W, λ) , and let a be an admissible slope.

- (1) Floer data for (ϕ, a) consists of a Hamiltonian $H_t : \widehat{W} \rightarrow \mathbb{R}$ and a family J_t of a $d\lambda$ -compatible almost complex structures on \widehat{W} satisfying the following conditions

$$\begin{aligned} H_{t+1} &= H_t \circ \phi, \\ J_{t+1} &= \phi^* J_t, \end{aligned}$$

and such that

$$\begin{aligned} H_t(x, r) &= ar, \\ J_t(x, r)\xi &= \xi, \\ J_t(x, r)\partial_r &= R, \end{aligned}$$

for $(x, r) \in \partial W \times (r_0, \infty)$ and $r_0 \in (0, \infty)$ large enough. Here, ξ and R stand for the contact structure and the Reeb vector field on ∂W , respectively.

(2) Floer data (H, J) for (ϕ, a) is said to be regular if

$$\det(d(\phi \circ \psi_1^H)(x) - \text{id}) \neq 0$$

for all fixed points x of $\phi \circ \psi_1^H$, and if the linearized operator of the Floer equation (2.3) below is surjective.

Let $\phi : W \rightarrow W$ be an exact symplectomorphism, let a be an admissible slope, and let (H, J) be regular Floer data for (ϕ, a) . The Floer homology $HF_*(W, \phi, H, J)$ is the Morse homology for the action functional

$$\mathcal{A}_{\phi, H} : \Omega_\phi \rightarrow \mathbb{R},$$

defined on the space of (ϕ) -twisted loops

$$\Omega_\phi := \left\{ \gamma : \mathbb{R} \rightarrow \widehat{W} : \phi(\gamma(t+1)) = \gamma(t) \right\}$$

by

$$\mathcal{A}_{\phi, H}(\gamma) := - \int_0^1 (\gamma^* \lambda + H_t(\gamma(t)) dt) - F_\phi(\gamma(1)).$$

In particular, the chain complex, denoted by $CF_*(W, \phi, H, J)$, is generated by critical points of $\mathcal{A}_{\phi, H}$, which happen to be the Hamiltonian twisted orbits

$$\text{Crit } \mathcal{A}_{\phi, H} = \{ \gamma \in \Omega_\phi : \dot{\gamma}(t) = X_{H_t}(\gamma(t)) \},$$

and the differential is obtained by counting unparametrized isolated solutions

$$u : \mathbb{R}^2 \rightarrow \widehat{W}$$

of the Floer equation

$$(2.3) \quad \partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0$$

with periodicity condition

$$\phi(u(s, t+1)) = u(s, t).$$

Different choices of regular Floer data for (ϕ, a) lead to canonically isomorphic Floer homologies. Hence the group $HF_*(W, \phi, a)$ is well defined. The group $HF_*(W, \text{id}, a)$ will also be denoted by $HF_*(W, a)$.

In the next two lemmas, we prove that the Floer homology $HF_*(W, \phi, a)$ remains the same if we change the Liouville form by adding an exact 1-form.

Lemma 2.5. *Let (W, λ) be a Liouville domain and let $f : W \rightarrow \mathbb{R}$ be a function. Then, a map $\phi : W \rightarrow W$ is an exact symplectomorphism of (W, λ) if, and only if, it is an exact symplectomorphism of $(W, \lambda + df)$.*

Proof. Denote the 1-form $\lambda + df$ by $\tilde{\lambda}$. Assume ϕ is an exact symplectomorphism of (W, λ) . Then, there exists a function $F : W \rightarrow \mathbb{R}$ such that

$$\phi^*\lambda - \lambda = dF,$$

and ϕ is compactly supported. Hence

$$\begin{aligned} \phi^*\tilde{\lambda} - \tilde{\lambda} &= \phi^*(\lambda + df) - \lambda - df \\ &= \phi^*\lambda - \lambda + d(f \circ \phi - f) \\ &= d(F + f \circ \phi - f), \end{aligned}$$

i.e. ϕ is an exact symplectomorphism of $(W, \tilde{\lambda})$. The opposite direction is proven analogously. \square

Lemma 2.6. *Let (W, λ) be a Liouville domain, let $\phi : W \rightarrow W$ be an exact symplectomorphism, and let $f : W \rightarrow \mathbb{R}$ be a function equal to 0 near the boundary. Then,*

$$HF_*(W, \lambda, \phi, a) \cong HF_*(W, \lambda + df, \phi, a),$$

for all admissible slopes a .

Proof. Denote by $\tilde{\lambda}$ the 1-form $\lambda + df$. Since λ and $\tilde{\lambda}$ agree near the boundary (where f is equal to 0), the slope of a Hamiltonian with respect to (W, λ) is the same as the one with respect to $(W, \tilde{\lambda})$. Moreover, the completions of (W, λ) and $(W, \tilde{\lambda})$ are symplectomorphic. Let H be a Hamiltonian with the slope equal to a . Denote by \tilde{F}_ϕ the compactly supported function $\tilde{W} \rightarrow \mathbb{R}$ such that

$$\phi^*\tilde{\lambda} - \tilde{\lambda} = d\tilde{F}_\phi.$$

The functions \tilde{F}_ϕ and F_ϕ are related by

$$\tilde{F}_\phi = F_\phi + f \circ \phi - f.$$

The lemma follows from the following sequence of equalities

$$\begin{aligned} \tilde{\mathcal{A}}_{\phi, H}(\gamma) &:= - \int_0^1 \left(\gamma^*\tilde{\lambda} + H_t(\gamma(t))dt \right) - \tilde{F}_\phi(\gamma(1)) \\ &= \mathcal{A}_{\phi, H}(\gamma) - \int_0^1 \frac{d}{dt}(f \circ \gamma)dt - f \circ \phi(\gamma(1)) + f(\gamma(1)) \\ &= \mathcal{A}_{\phi, H}(\gamma) - f(\gamma(1)) + f(\gamma(0)) - f \circ \phi(\gamma(1)) + f(\gamma(1)) \\ &= \mathcal{A}_{\phi, H}(\gamma), \end{aligned}$$

for $\gamma \in \Omega_\phi$. \square

The graded group $CF_*^{<c}(W, \phi, H, J)$, $c \in \mathbb{R}$, generated by the elements of $\text{Crit } \mathcal{A}_{\phi, H}$ having the action less than c is a subcomplex of $CF_*(W, \phi, H, J)$. It fits into a short exact sequence of chain complexes

$$0 \rightarrow CF_*^{<c}(W, \phi, H, J) \rightarrow CF_*(W, \phi, H, J) \rightarrow CF_*^{\geq c}(W, \phi, H, J) \rightarrow 0,$$

where

$$CF_*^{\geq c}(W, \phi, H, J) := \frac{CF_*(W, \phi, H, J)}{CF_*^{<c}(W, \phi, H, J)}.$$

The homologies of the chain complexes $CF_*^{<c}(\dots)$ and $CF_*^{\geq c}(\dots)$ are denoted by $HF_*^{<c}(\dots)$ and $HF_*^{\geq c}(\dots)$, respectively.

For more details on Floer homology for an exact symplectomorphism, see [7].

2.3. Notation. Throughout, (W_2, λ) is a $2n$ -dimensional Liouville domain with connected boundary, $W_1 \subset W_2$ is a codimension-0 submanifold with connected boundary such that (W_1, λ) is a Liouville domain in its own right, and $\phi : W_1 \rightarrow W_1$ is an exact symplectomorphism.

3. TRANSFER MORPHISM

The transfer morphism is essentially the map

$$HF_*(W_2, \phi, H, J) \rightarrow HF_*^{\geq 0}(W_2, \phi, H, J)$$

induced by the natural projection of chain complexes $CF_*(\dots) \rightarrow CF_*^{\geq 0}(\dots)$ for a Hamiltonian H that is C^2 -close to a so-called stair-like Hamiltonian. The group $HF_*(W_2, \phi, H, J)$ is isomorphic to $HF_*(W_2, \phi, a)$, where a is the slope of H , and $HF_*^{\geq 0}(W_2, \phi, H, J)$ can be identified with $HF_*(W_1, \phi, b)$ for a certain slope b . In the rest of the section, we describe the construction in more details and prove Theorem 1.1.

3.1. Stair-like Hamiltonians.

Definition 3.1. Let $0 < a < b$ be such that a is admissible with respect to W_2 and b is admissible with respect to W_1 , and let b_0 be the greatest period of some Reeb orbit on ∂W_1 that is smaller than b . The set $\mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b)$ is defined to be the set of (time-independent) Hamiltonians $H : \widehat{W}_2 \rightarrow \mathbb{R}$ having the following property. There exist positive real numbers $\delta_1, \delta_2, \delta_3, A, B, C \in (0, \infty)$ and functions

$$\begin{aligned} h_1 : [\delta_1, 2\delta_1] &\rightarrow \mathbb{R}, \\ h_2 : [1 - \delta_2, 1] &\rightarrow \mathbb{R}, \\ h_3 : [1 + \delta_3, 1 + 2\delta_3] &\rightarrow \mathbb{R}, \end{aligned}$$

such that

- (1) ϕ is compactly supported in the interior of W^{δ_1} ,
- (2) $\sup_{p \in W_1} |F_\phi(p)| < A$,
- (3) h_1 and h_3 are convex and strictly increasing,
- (4) h_2 is concave, strictly increasing, and $h_2(r) > rb_0$ for all $r \in [1 - \delta_2, 1]$

and such that

$$H(p) = \begin{cases} -A & \text{for } p \in W_1^\delta, \\ h_1(r) & \text{for } p = (x, r) \in \partial W_1 \times [\delta_1, 2\delta_1], \\ b(r - 2\delta_1) & \text{for } p = (x, r) \in \partial W_1 \times [2\delta_1, 1 - \delta_2), \\ h_2(r) & \text{for } p = (x, r) \in \partial W_1 \times [1 - \delta_2, 1], \\ B & \text{for } p \in W_2^{1+\delta_3} \setminus W_1, \\ h_3(r) & \text{for } p = (x, r) \in \partial W_2 \times (1 + \delta_3, 1 + 2\delta_3), \\ ar + C & \text{for } p = (x, r) \in \partial W_2 \times [1 + 2\delta_3, \infty). \end{cases}$$

A typical element of $\mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b)$ is shown in Figure 1.

Definition 3.2. Let W be a Liouville domain, and let $\phi : \widehat{W} \rightarrow \widehat{W}$ be a diffeomorphism. The **support radius** $\rho(W, \phi)$ of ϕ is defined by

$$\rho(W, \phi) := \inf \{r \in (0, \infty) : \text{supp } \phi \subset W^r\}.$$

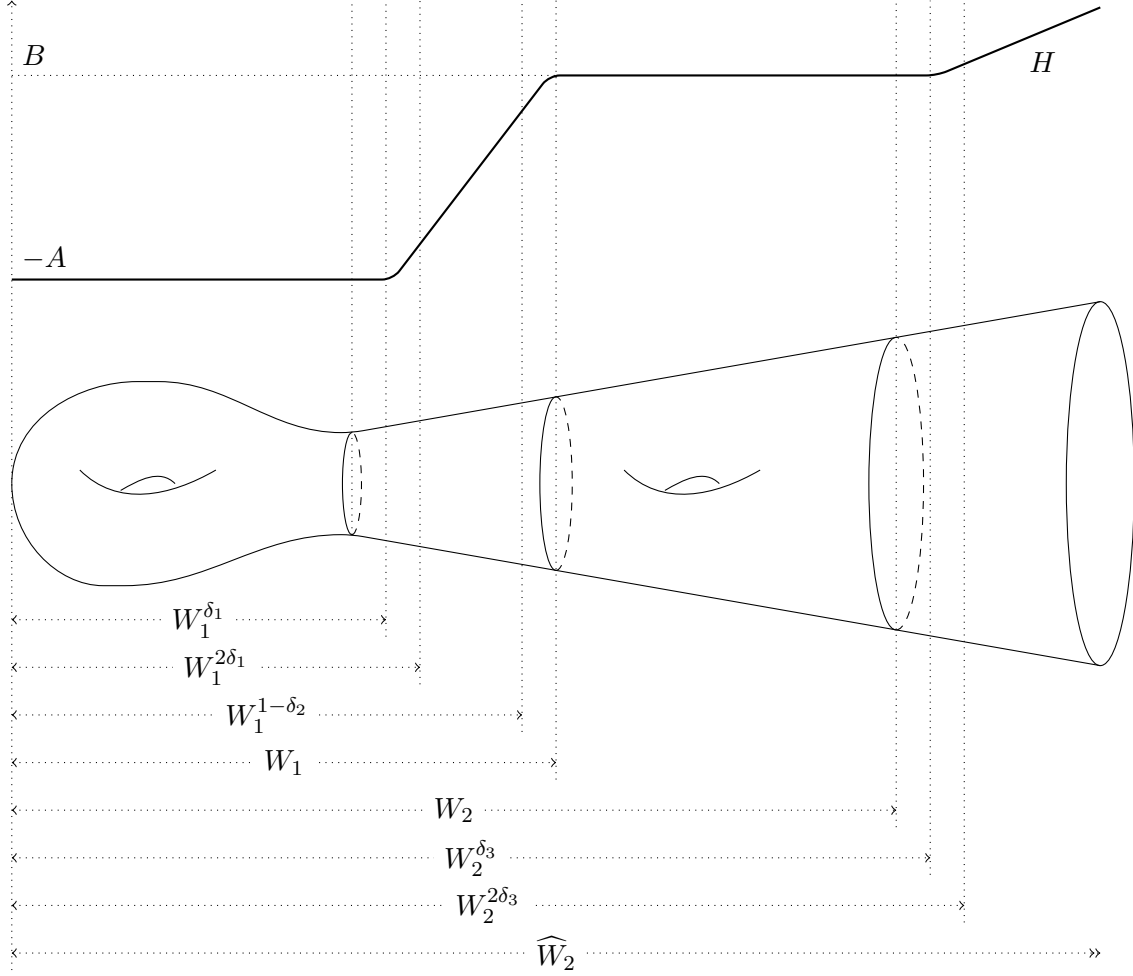


FIGURE 1. A stair-like Hamiltonian

Lemma 3.3. *Let a, b , and b_0 be as in Definition 3.1. If*

$$C(W_1, \phi) < \min \{b - b_0, b - a\},$$

where

$$(3.4) \quad C(W_1, \phi) := 2 \max \left\{ \sup_{p \in W_1} |F_\phi(p)|, \rho(W_1, \phi) \right\},$$

then the set $\mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b)$ is not empty.

Proof. The constants $\delta_1, \delta_2, \delta_3, A, B, C$ from Definition 3.1 are chosen in the following way and the following order.

- Choose

$$A \in \left(\frac{C(W_1, \phi)}{2}, \frac{\min\{b - b_0, b - a\}}{2} \right),$$

- choose $\delta_1 \in \left(\frac{A}{b}, \frac{\min\{b-b_0, b-a\}}{2b}\right)$,
- choose $\delta_2 \in \left(0, 1 - \frac{2b\delta_1}{b-b_0}\right)$,
- choose $B \in (\max\{a, b_0, b - 2b\delta_1 - b\delta_2\}, b - 2b\delta_1)$,
- choose $\delta_3 \in \left(0, \frac{B-a}{a}\right)$,
- choose $C \in (\max\{0, B - a(1 + 2\delta_3)\}, B - a(1 + \delta_3))$.

It follows that the following inequalities hold

$$(3.5) \quad 0 < -A - (2b\delta_1) - b\delta_1 < b\delta_1,$$

$$(3.6) \quad 0 < B - (-2b\delta_1) - b(1 - \delta_2) < b\delta_2,$$

$$(3.7) \quad b(1 - \delta_2) + (-2b\delta_1) > (1 - \delta_2)b_0,$$

$$(3.8) \quad B > b_0,$$

$$(3.9) \quad 0 < B - C - a(1 + \delta_3) < a\delta_3.$$

Due to Lemma A.1 and Lemma A.2, (3.5) implies that h_1 exists, (3.6), (3.7), (3.8) imply that h_2 exists, and (3.9) implies that h_3 exists. This finishes the proof. \square

Lemma 3.4. *Let $\phi, a, b, H, \delta_1, \delta_2, \delta_3, A, B, C$ be as in Definition 3.1. The ranges of the action functional $\mathcal{A}_{\phi, H}$ when evaluated on ϕ -twisted Hamiltonian orbits contained in the regions*

$$\begin{aligned} I &:= W_1^{\delta_1}, \quad II := \partial W_1 \times (\delta_1, 2\delta_1), \quad III := \partial W_1 \times (1 - \delta_2, 1), \\ IV &:= W_2^{1+\delta_3} \setminus \text{int } W_1, \quad V := \partial W_2 \times (1 + \delta_3, 1 + 2\delta_3) \end{aligned}$$

are given in the following table.

region	I	II	III	IV	V
$\mathcal{A}_{\phi, H} \in$	$[A - \ F_\phi\ , A + \ F_\phi\]$	$(A, 2b\delta_1)$	$(-B, 0)$	$\{-B\}$	$(-B, -C)$

Note that each ϕ -twisted orbit of H is contained in one of these regions.

Proof. We will prove the statement only for region III. For the other regions, the proof is similar and even more direct.

The symplectomorphism ϕ is equal to the identity in region III. Therefore, the twisted Hamiltonian orbits coincide with Hamiltonian loops. Additionally, due to the form of the Hamiltonian H in this region, they can be explicitly described in terms of periodic Reeb orbits on ∂W_1 . Each 1-periodic Hamiltonian orbit in region III is given by

$$(3.10) \quad t \mapsto (\gamma(-h'_2(r_0)t), r_0)$$

where $\gamma : \mathbb{R} \rightarrow \partial W_1$ is a $h'_2(r_0)$ -periodic Reeb orbit. The action of (3.10) is equal to

$$r_0 h'_2(r_0) - h_2(r_0).$$

Consider the function

$$f : [1 - \delta_2, 1] \rightarrow \mathbb{R} : r \mapsto r h'_2(r) - h_2(r).$$

Since $f'(r) = r h''_2(r) \leq 0$, f is decreasing. Hence the action of a 1-periodic Hamiltonian orbit in region III lies in the interval $[f(1), f(r_1)] = (-B, f(r_1))$, where $r_1 \in (1 - \delta_2, 1)$ is

the smallest number such that $h'_2(r_1)$ is a period of some Reeb orbit on ∂W_1 . Since h'_2 is continuous and decreasing (h_2 is concave), we get $h'_2(r_1) = b_0$. Therefore,

$$f(r_1) = r_1 b_0 - h_2(r_1) < 0$$

(because, by definition, $h_2(r) > r b_0$ for all $r \in [1 - \delta_2, 1]$). This finishes the proof. \square

Proposition 3.5. *Let $\phi : W_1 \rightarrow W_1$ be an exact symplectomorphism and let $\phi_t : W_1 \rightarrow W_1$ be the isotopy of exact symplectomorphisms given by*

$$\phi_t := \left(\psi_t^\lambda \right)^{-1} \circ \phi \circ \psi_t^\lambda,$$

where $\psi_t^\lambda : \widehat{W}_1 \rightarrow \widehat{W}_1$ is the Liouville flow. Then,

$$F_{\phi_t} = e^{-t} F_\phi \circ \psi_t^\lambda \quad \text{and} \quad \rho(\phi_t, W_1, \lambda) = e^{-t} \rho(\phi, W_1, \lambda).$$

Consequently, $C(W_1, \phi_t) = e^{-t} C(W_1, \phi)$.

Proof. Denote ψ_t^λ by ψ_t for simplicity. The Cartan formula implies

$$\psi_t^* \lambda = e^t \lambda \quad \text{and} \quad (\psi_t^{-1})^* \lambda = e^{-t} \lambda.$$

Therefore

$$\begin{aligned} dF_{\phi_t} &= \phi_t^* \lambda - \lambda = (\psi_t^{-1} \circ \phi \circ \psi_t)^* \lambda - \lambda = \psi_t^* \phi^* (\psi_t^{-1})^* \lambda - \lambda \\ &= e^{-t} \psi_t^* (\phi^* \lambda - \lambda) = e^{-t} d(F_\phi \circ \psi_t). \end{aligned}$$

Hence $F_{\phi_t} = e^{-t} F_\phi \circ \psi_t$. Since

$$\text{supp } \phi_t \subset W^r \iff \text{supp } \phi \subset W^{e^t r},$$

the equality $\rho(W_1, \phi_t) = e^{-t} \rho(W_1, \phi)$ holds. \square

3.2. Construction.

Definition 3.6. Let a and b be as in Definition 3.1. **Transfer data** for (ϕ, W_1, W_2, a, b) is Floer data (H, J) for $(\phi : W_2 \rightarrow W_2, b)$ satisfying the following. There exists a Hamiltonian $\overline{H} \in \mathcal{H}_{\text{stair}}(\phi, W_1, W_2, a, b)$ such that $H = \overline{H}$ on $\partial W_1 \times [2\delta_1, 1 - \delta_2]$ and $\partial W_2 \times [1 + 2\delta_3, \infty]$, and such that the action functional is positive at twisted Hamiltonian orbits in $W_1^{2\delta_1}$ and negative at the other twisted orbits (this holds for H that is C^2 -close to \overline{H}). Additionally,

$$dr \circ J_t = -\lambda$$

in $\partial W_1 \times [2\delta_1, 1 - \delta_2]$.

Lemma 3.7. *Let (H, J) be a regular transfer data for (ϕ, W_1, W_2, a, b) . By definition, there exist $\delta_1, \delta_2 \in (0, 1)$ such that $2\delta_1 + \delta_2 < 1$ and $H_t(x, r) = b(r - 2\delta_1)$ for all $(x, r) \in \partial W_1 \times [2\delta_1, 1 - \delta_2]$. Let $G_t : \widehat{W}_1 \rightarrow \mathbb{R}$ be the Hamiltonian defined by*

$$G_t(p) := \begin{cases} H_t(p) & \text{for } p \in W_1^{1-\delta_2} \\ b(r - 2\delta_1) & \text{for } p \in \partial W_1 \times [2\delta_1, \infty). \end{cases}$$

Then,

$$HF_*(W_1, \phi, G, J) = HF_*^{\geq 0}(W_2, \phi, H, J).$$

Proof. The generators of the chain complexes $CF_*(W_1, \phi, G, J)$ and $CF_*^{\geq 0}(W_2, \phi, H, J)$ coincide. Therefore it is enough to prove the following. The solutions

$$u : \mathbb{R}^2 \rightarrow \widehat{W}_2, \quad \phi(u(s, t + 1)) = u(s, t)$$

of the Floer equation which connect two Hamiltonian twisted orbits in W_1 satisfy $u(\mathbb{R}^2) \subset W_1$. This follows from the part a) of Theorem 4.5 in [2]. See also [2, Proposition 4.4]. \square

We are going to define the transfer morphism in couple of steps. Definition 3.8 defines it for positive slopes with additional technical conditions. Definition 3.9 and Definition 3.12 eliminate the technical conditions. And, finally, Proposition 3.11 extends the transfer morphism to the case of infinite slopes.

Definition 3.8. Let $a, b \in (0, \infty)$ be positive real numbers such that a and b are admissible with respect to W_2 and W_1 , respectively, and such that $a < b$. Let b_0 be the greatest period of some Reeb orbit on ∂W_1 that is smaller than b . Assume

$$C(W_1, \phi) < \min\{b - b_0, b - a\}.$$

The **transfer morphism** is defined to be the map

$$HF_*(W_2, \phi, a) \rightarrow HF_*(W_1, \phi, b),$$

induced by the natural map of chain complexes

$$CF_*(W_2, \phi, H, J) \rightarrow CF_*^{\geq 0}(W_2, \phi, H, J),$$

where (H, J) is regular Transfer data for (ϕ, W_1, W_2, a, b) .

Definition 3.9. Let $a \in (0, \infty)$ be admissible with respect to W_2 , and let $b \in (0, \infty)$ be admissible with respect to W_1 . Assume $a < b$. The transfer morphism is defined as the composition

$$\begin{array}{ccc} HF_*(W_2, \phi, a) & \dashrightarrow & HF_*(W_1, \phi, b) \\ I(\{\phi_t\}) \downarrow & & \uparrow I(\{\phi_t\})^{-1} \\ HF_*(W_2, \phi_1, a) & \xrightarrow{T} & HF_*(W_1, \phi_1, b). \end{array}$$

Here,

$$\phi_t := (\psi_{ct}^\lambda)^{-1} \circ \phi \circ \psi_{ct}^\lambda, \quad t \in [0, 1],$$

for $c \in (0, \infty)$ such that

$$C(W_1, \phi_1) < \min\{b - b_0, b - a\},$$

T is the transfer morphism of Definition 3.8, and $I(\{\phi_t\})$ is the isomorphism furnished by isotopy $\{\phi_t\}$ (see Section 2.8 in [7]).

Lemma 3.10. *In the situation of Definition 3.9, the transfer morphism does not depend on the choice of c .*

Proof. Let c_1 and c_2 be positive real numbers such that

$$C(W_1, \phi_i) < \min\{b - b_0, b - a\}, \quad i \in \{1, 2\},$$

where

$$\phi_i := (\psi_{c_i}^\lambda)^{-1} \circ \phi \circ \psi_{c_i}^\lambda, \quad i \in \{1, 2\}.$$

It is enough to prove that the diagram

$$\begin{array}{ccc}
 HF_*(W_2, \phi_1, a) & \longrightarrow & HF_*(W_1, \phi_1, b) \\
 \uparrow & & \downarrow \\
 HF_*(W_2, \phi, a) & & HF_*(W_1, \phi, b) \\
 \downarrow & & \uparrow \\
 HF_*(W_2, \phi_2, a) & \longrightarrow & HF_*(W_1, \phi_2, b)
 \end{array}$$

commutes, where the vertical arrows stand for the isomorphisms induced by symplectic isotopies. These isomorphisms, as it follows from their definition in [7, Section 2.8], coincide with naturality isomorphisms (see Section 2.7 in [7]) with respect to some Hamiltonians that are compactly supported in the interior of W_1 . Hence, it is enough to prove that the diagram

$$\begin{array}{ccc}
 HF(W_2, \phi_1, a) & \longrightarrow & HF(W_1, \phi_1, b) \\
 \mathcal{N}(H) \downarrow & & \downarrow \mathcal{N}(K) \\
 HF(W_2, \phi_2, a) & \longrightarrow & HF(W_1, \phi_2, b)
 \end{array}$$

commutes, where $H, K : \widehat{W}_2 \rightarrow \mathbb{R}$ are compactly supported in the interior of W_1 , and $\mathcal{N}(H), \mathcal{N}(K)$ denote the corresponding naturality isomorphisms. This follows from the following fact. The naturality with respect to a Hamiltonian $\widehat{W}_2 \rightarrow \mathbb{R}$ that is compactly supported in the interior of W_1 converts transfer data for (ϕ_1, W_1, W_2, a, b) into transfer data for (ϕ_2, W_1, W_2, a, b) . \square

Proposition 3.11. *The diagram*

$$\begin{array}{ccc}
 HF_*(W_2, \phi, a) & \longrightarrow & HF_*(W_1, \phi, b) \\
 \downarrow & & \downarrow \\
 HF_*(W_2, \phi, a') & \longrightarrow & HF_*(W_1, \phi, b'),
 \end{array}$$

consisting of transfer morphisms and continuation maps, commutes whenever $a, b, a', b' \in (0, \infty)$ are such that all the maps in the diagram are well defined.

Proof of Theorem 1.1. The proof follows from the proof of Proposition 4.7 in [2]. \square

Definition 3.12. Let $a \in (0, \infty)$ be admissible with respect to both W_1 and W_2 . Let $\varepsilon > 0$ be a positive number such that the elements of $[a, a + \varepsilon]$ are all admissible with respect to W_1 . The transfer morphism

$$HF_*(W_2, \phi, a) \rightarrow HF_*(W_1, \phi, a)$$

is defined to be the composition of the transfer morphism

$$HF_*(W_2, \phi, a) \rightarrow HF_*(W_1, \phi, a + \varepsilon)$$

of Definition 3.9 and the inverse of the continuation map

$$\Phi^{-1} : HF_*(W_1, \phi, a + \varepsilon) \rightarrow HF_*(W_1, \phi, a).$$

Lemma 3.13. *The transfer morphism of Definition 3.12 is well defined and does not depend on the choice of ε .*

Proof. Lemma A.3 implies that ε from Definition 3.12 exists. Due to [7, Lemma 2.25], the continuation map

$$\Phi : HF_*(W_1, \phi, a) \rightarrow HF_*(W_1, \phi, a + \varepsilon)$$

is an isomorphism. Hence, its inverse Φ^{-1} is well defined. Independence of the choice of ε follows from Proposition 3.11. \square

Proposition 3.11 enables us to extend the transfer morphism to the case of infinite slopes. The proposition is still true after the extensions of the transfer morphism.

Proof of Theorem 1.1. The existence of the transfer morphism has already been shown above. It remains to prove that the diagram (1.2) commutes. This follows from the proof of Proposition 4.7 in [2]. \square

4. APPLICATIONS

4.1. Iterated ratio.

Definition 4.1. Let $\phi : W \rightarrow W$ be an exact symplectomorphism of a Liouville domain W . The **iterated ratio** $\kappa(W, \phi)$ is defined to be the number

$$\kappa(W, \phi) := \limsup_{m \rightarrow \infty} \frac{\dim HF(W, \phi^m, \varepsilon)}{m}.$$

Here $\varepsilon > 0$ is small enough (smaller than any positive period of some Reeb orbit on ∂W).

Remark 4.2. The iterated ratio is invariant under compactly supported symplectic isotopies. It measures the linear growth rate of $\dim HF(W, \phi^m, \varepsilon)$. There is yet another similar invariant, the Floer theoretic entropy,

$$h_{\text{Floer}}(W, \phi) := \limsup_{m \rightarrow \infty} \frac{\log(\dim HF(W, \phi^m, \varepsilon))}{m},$$

which measures the exponential growth rate of $\dim HF(W, \phi, \varepsilon)$. If W is a surface and ϕ its area-preserving diffeomorphism that has a pseudo-Anosov component, then $h_{\text{Floer}}(W, \phi) > 0$ [6, page 167]. Consequently, $\kappa(W, \phi)$ is equal to infinity.

Proof of Theorem 1.2. Let $\varepsilon > 0$ be small. It is enough to prove that the number

$$|\dim HF(W_1, \phi^m, \varepsilon) - \dim HF(W_2, \phi^m, \varepsilon)|$$

is bounded (by a constant not depending on m). Let $\psi_m : W_1 \rightarrow W_1$ be an exact symplectomorphism that is isotopic to ϕ^m through exact symplectomorphisms, and such that $C(W_1, \psi_m) < \frac{\varepsilon}{2}$. Such a symplectomorphism exists for each $m \in \mathbb{N}$ due to Proposition 3.5. Choose transfer data (H^m, J^m) for $(\psi_m, W_1, W_2, \frac{\varepsilon}{2}, \varepsilon)$, such that $H^m = H^1$ and $J^m = J^1$ on $W_2 \setminus W_1$. The short exact sequence

$$0 \rightarrow CF_*^{<0}(W_2, \psi_m, H^m, J^m) \rightarrow CF_*(W_2, \psi_m, H^m, J^m) \rightarrow CF_*^{\geq 0}(W_2, \psi_m, H^m, J^m) \rightarrow 0$$

of chain complexes induces the long exact sequence in homology

$$\cdots \rightarrow HF_*^{<0}(W_2, \psi_m, H^m, J^m) \rightarrow HF_*(W_2, \psi_m, H^m, J^m) \rightarrow HF_*^{\geq 0}(W_2, \psi_m, H^m, J^m) \rightarrow \cdots$$

By applying identifications as in Section 3.2, the long exact sequence is transformed into

$$(4.11) \quad \cdots \rightarrow HF_*^{<0}(W_2, \psi_m, H^m, J^m) \rightarrow HF_*\left(W_2, \psi_m, \frac{\varepsilon}{2}\right) \rightarrow HF_*(W_1, \psi_m, \varepsilon) \rightarrow \cdots$$

The chain complexes $CF_*^{<0}(W_2, \psi_m, H^m, J^m)$, $m \in \mathbb{N}$ are generated by twisted orbits contained in $W_2 \setminus W_1$. In this region $\psi_m = \text{id}$ and $(H^m, J^m) = (H^1, J^1)$. Hence

$$CF_*^{<0}(W_2, \psi_m, H^m, J^m), \quad m \in \mathbb{N}$$

are generated by the same set of generators. This does not imply that

$$HF_*^{<0}(W_2, \psi_m, H^m, J^m), \quad m \in \mathbb{N}$$

are isomorphic, because the differentials may differ. However,

$$\dim HF_*^{<0}(W_2, \psi_m, H^m, J^m) \leq \dim CF_*^{<0}(W_2, \psi_m, H^m, J^m) = \dim CF_*^{<0}(W_2, \psi_1, H^1, J^1),$$

and the number on the right-hand side does not depend on m . The long exact sequence (4.11) implies

$$\left| \dim HF\left(W_2, \psi_m, \frac{\varepsilon}{2}\right) - \dim HF(W_1, \psi_m, \varepsilon) \right| < 2 \dim CF^{<0}(W_2, \psi_1, H^1, J^1).$$

Finally, note that

$$HF\left(W_2, \psi_m, \frac{\varepsilon}{2}\right) \cong HF\left(W_2, \phi^m, \frac{\varepsilon}{2}\right) \cong HF(W_2, \phi^m, \varepsilon),$$

and

$$HF(W_1, \psi_m, \varepsilon) \cong HF(W_1, \phi^m, \varepsilon).$$

This finishes the proof. \square

The following proposition shows that symplectomorphisms which are supported in the cylindrical part of a Liouville domain have finite iterated ratio. This proposition will not be used in the rest of the paper.

Proposition 4.3. *Let (W, λ) be a Liouville domain, and let $\phi : W \rightarrow W$ be an exact symplectomorphism that is compactly supported in $\partial W \times (0, 1)$. Then*

$$\kappa(W, \phi) < \infty.$$

Proof. The idea of the proof is the following. We construct an exact symplectomorphism $\psi_m : W \rightarrow W$ that is symplectically isotopic to ϕ^m relative to the boundary, and such that it consists of m “copies” of ϕ with disjoint supports. By choosing a suitable Hamiltonian, one can ensure that each “copy” contributes the same number of generators to the Floer chain complex. Since the number of “copies” grows linearly with respect to m , the iterated ratio has to be smaller than the number of generators furnished by a single “copy.”

Now, we implement the idea rigorously. By the assumption, there exists $\delta \in (0, \frac{1}{2})$ such that ϕ is compactly supported in $\partial W \times (2\delta, 1)$. Let $H_t : \widehat{W} \rightarrow \mathbb{R}$ be a Hamiltonian equal to

$$(x, r) \mapsto \varepsilon r$$

on $\partial W \times (\delta, 2\delta) \cup \partial W \times (1, \infty)$ such that

$$H_{t+1} = H_t \circ \phi,$$

and such that

$$\det(d(\phi \circ \psi_1^H)(x) - \text{id}) \neq 0$$

for all fixed points x of $\phi \circ \psi_1^H$. The non-degeneracy condition implies that there are finitely many Hamiltonian twisted orbits. We denote by A and B the numbers of such orbits in W^δ and $\partial W \times (2\delta, 1)$, respectively.

Fix $m \in \mathbb{N}$. Denote by $\varphi_0 : W \rightarrow W$ the symplectomorphism

$$\varphi_0 := \psi_T^\lambda \circ \phi \circ \left(\psi_T^\lambda\right)^{-1},$$

where $T := (m-1)\ln \delta - m \ln(1+\delta)$, and ψ_t^λ is the Liouville flow. Let $\varphi_j, j \in \{1, \dots, m-1\}$ be the symplectomorphism given by

$$\varphi_j := \psi_{jc}^\lambda \circ \varphi_0 \circ \left(\psi_{jc}^\lambda\right)^{-1},$$

where $c := \ln(1+\delta) - \ln \delta$. Similarly, denote

$$\begin{aligned} H_t^0 &:= H_t \circ \left(\psi_T^\lambda\right)^{-1}, \\ H_t^j &:= H_t^0 \circ \left(\psi_{jc}^\lambda\right)^{-1}, \quad j \in \{1, \dots, m-1\}. \end{aligned}$$

The symplectomorphisms $\varphi_0, \varphi_1, \dots, \varphi_{m-1}$ are all symplectically isotopic to ϕ relative to the boundary, and they are compactly supported in

$$(4.12) \quad \partial W \times (r_0, r_0 e^c), \quad \partial W \times (r_0 e^c, e_0 e^{2c}), \quad \dots, \quad \partial W \times (r_0 e^{(m-1)c}, r_0 e^{mc})$$

respectively. Here,

$$r_0 := \frac{\delta^m}{(1+\delta)^m},$$

and c is as above. (It follows that $r_0 e^{mc} = 1$ and consequently the sets (4.12) are all subsets of W . Hence, φ_j is compactly supported in the interior of W for all $j \in \{0, \dots, m-1\}$.) Let $\psi_m : W \rightarrow W$ be the exact symplectomorphism

$$\psi_m := \varphi_0 \circ \dots \circ \varphi_{m-1},$$

and let $G_t : \widehat{W} \rightarrow \mathbb{R}$ be the Hamiltonian defined by

$$G_t(p) := \begin{cases} H_t^0(p) & \text{for } p \in W^{r_0}, \\ H_t^j(p) & \text{for } p \in \partial W \times (r_0 e^{(j-1)c}, r_0 e^{jc}), j \in \{1, \dots, m-1\} \\ \varepsilon r & \text{otherwise.} \end{cases}$$

By construction, G_t has exactly $A + mB$ Hamiltonian ψ_m -twisted loops, and they are all non-degenerate. This implies

$$\dim HF(W, \phi^m, \varepsilon) = \dim HF(W, \psi_m, \varepsilon) \leq A + mB.$$

The first equality follows from ϕ^m being symplectically isotopic to ψ_m relative to the boundary. Hence,

$$\kappa(W, \phi) \leq \lim_{m \rightarrow \infty} \frac{A + mB}{m} = B < \infty,$$

and the proof is finished. \square

4.2. Examples. In this section, we compute the iterated ratio for the fibered Dehn twist $\tau : D^*\mathbb{S}^n \rightarrow D^*\mathbb{S}^n$ [5, 1] of the disk cotangent bundle of the sphere \mathbb{S}^n endowed with the standard Riemannian metric. The Floer homology $HF(D^*\mathbb{S}^n, \tau^m, \varepsilon)$, $m \in \mathbb{N}$, is isomorphic (as a \mathbb{Z}_2 -vector space) to $HF(D^*\mathbb{S}^n, \text{id}, 2\pi m + \varepsilon)$. This follows from Proposition 2.30 and Remark 3.2 in [7]. Hence

$$\kappa(D^*\mathbb{S}^n, \tau) = \lim_{m \rightarrow \infty} \frac{\dim HF(D^*\mathbb{S}^n, \text{id}, 2\pi m + \varepsilon)}{m}.$$

Proposition 4.4. *Let m and n be positive integers with $n > 2$, let $\varepsilon \in (0, 2\pi)$, and let $D^*\mathbb{S}^n$ be the disk cotangent bundle of \mathbb{S}^n with respect to the standard Riemannian metric. Then,*

$$HF_k(D^*\mathbb{S}^2, 2\pi m + \varepsilon) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k \in \{0, 1, 2m+1, 2m+2\}, \\ \mathbb{Z}_2^2 & \text{for } k \in \mathbb{N} \text{ and } 2 \leq k \leq 2m, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$HF_k(D^*\mathbb{S}^n, 2\pi m + \varepsilon) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k \in \{\ell(n-1), \ell(n-1)+n : \ell \in \mathbb{Z} \text{ \& } 0 \leq \ell \leq 2m\}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\dim HF_k(D^*\mathbb{S}^n, 2\pi m + \varepsilon) = \dim HF_k(D^*\mathbb{S}^2, 2\pi m + \varepsilon) = 4m + 2.$$

Proof. We will consider only the case of \mathbb{S}^2 . The proof for other cases is analogous and even simpler.

The Reeb flow on $S^*\mathbb{S}^2 := \partial D^*\mathbb{S}^2$ is periodic with minimal period equal to 2π . Hence, there exists a homology long exact sequence

$$(4.13) \quad \begin{array}{c} \cdots \longrightarrow HF_k(D^*\mathbb{S}^2, 2\pi\ell + \varepsilon) \longrightarrow HF_k(D^*\mathbb{S}^2, 2\pi(\ell+1) + \varepsilon) \longrightarrow H_{k+\Delta_{\ell+1}}(S^*\mathbb{S}^2) \\ \hspace{10em} \uparrow \\ \hspace{10em} \hookrightarrow HF_{k-1}(D^*\mathbb{S}^2, 2\pi\ell + \varepsilon) \longrightarrow \cdots, \end{array}$$

where $\ell \in \mathbb{N} \cup \{0\}$, and Δ_ℓ is a shift in grading [4, Lemma 3.6]. The shift Δ_ℓ is equal to $-(2\ell-1)$ [3, Proposition 5.12]. The long exact sequence (4.13) implies

$$HF_k(D^*\mathbb{S}^2, 2\pi\ell + \varepsilon) \xrightarrow{\cong} HF_k(D^*\mathbb{S}^2, 2\pi(\ell+1) + \varepsilon),$$

for $k < 2\ell$. Therefore

$$(4.14) \quad HF_k(D^*\mathbb{S}^2, 2\pi\ell + \varepsilon) \cong SH_k(D^*\mathbb{S}^2), \quad \text{for } k < 2\ell.$$

We prove the proposition by induction. Using (4.13) with $\ell = 0$, we get

$$\begin{aligned} \dim HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon) &= \dim HF_3(D^*\mathbb{S}^2, 2\pi + \varepsilon) + 1, \\ HF_4(D^*\mathbb{S}^2, 2\pi + \varepsilon) &\cong \mathbb{Z}_2 \\ HF_k(D^*\mathbb{S}^2, 2\pi + \varepsilon) &= 0, \quad \text{for } k \notin \{0, 1, 2, 3, 4\}, \\ \dim HF_3(D^*\mathbb{S}^2, 2\pi + \varepsilon) &\leq 1. \end{aligned}$$

Consequently,

$$(4.15) \quad \dim HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon) \leq 2.$$

The equation (4.14) implies

$$\begin{aligned} HF_0(D^*\mathbb{S}^2, 2\pi + \varepsilon) &\cong SH_0(D^*\mathbb{S}^2) \cong \mathbb{Z}_2, \\ HF_1(D^*\mathbb{S}^2, 2\pi + \varepsilon) &\cong SH_1(D^*\mathbb{S}^2) \cong \mathbb{Z}_2, \\ HF_2(D^*\mathbb{S}^2, 4\pi + \varepsilon) &\cong SH_2(D^*\mathbb{S}^2) \cong \mathbb{Z}_2^2. \end{aligned}$$

(The computation of the symplectic homology for $D^*\mathbb{S}^2$ can be done by using [9] and [10, page 21].) Consider the segment

$$\cdots \longrightarrow HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon) \longrightarrow HF_2(D^*\mathbb{S}^2, 4\pi + \varepsilon) \longrightarrow 0$$

of the long exact sequence (4.13) with $\ell = 1$. It implies

$$\dim HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon) \geq \dim HF_2(D^*\mathbb{S}^2, 4\pi + \varepsilon) = 2.$$

Therefore (see (4.15))

$$HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon) \cong \mathbb{Z}_2^2 \quad \text{and} \quad HF_3(D^*\mathbb{S}^2, 2\pi + \varepsilon) \cong \mathbb{Z}_2$$

This proves the basis of the induction. Assume now the claim holds for $1, \dots, m$. The groups $HF_k(D^*\mathbb{S}^2, 2\pi(m+1) + \varepsilon)$ for $k \in \{0, \dots, 2m-1\} \cup \{2m+4\}$ can be computed directly from (4.13) with $\ell = m$. The isomorphism (4.14) computes the groups $HF_{2m}(\cdots)$ and $HF_{2m+1}(\cdots)$. Finally, the groups $HF_{2m+2}(\cdots)$ and $HF_{2m+3}(\cdots)$ are computed in the same way as $HF_2(D^*\mathbb{S}^2, 2\pi + \varepsilon)$ and $HF_3(D^*\mathbb{S}^2, 2\pi + \varepsilon)$ in the basis of the induction. (We are not going to repeat the argument here.) This finishes the proof. \square

Proof of Theorem 1.3. By the Weinstein neighbourhood theorem, there exists a neighbourhood V of L in W that is symplectomorphic to the disk-cotangent bundle $D_\rho^*\mathbb{S}^n$ for small enough radius $\rho \in (0, \infty)$. The manifold V is not, in general, a Liouville domain when considered with the 1-form λ . Namely, the Liouville vector field X_λ might not be pointing out on the boundary ∂V . There is, however, a 1-form λ_0 furnished by the canonical Liouville form on $D_\rho^*\mathbb{S}^n$ that makes V into a Liouville domain. The restriction $(\lambda - \lambda_0)|_L = \lambda|_L$ is an exact form because L is an exact Lagrangian submanifold. This, together with the inclusion $L \hookrightarrow V$ being homotopy equivalence, implies that there exists a function $f : V \rightarrow \mathbb{R}$ such that $\lambda - \lambda_0 = df$. Let $\chi : V \rightarrow [0, 1]$ be a function equal to 0 on $V^{\frac{1}{3}}$ and to 1 on the complement of $V^{\frac{2}{3}}$. Consider the Liouville form

$$\lambda_1 = \lambda_0 + d(\chi f).$$

Note that λ_1 is equal to λ_0 on $V^{\frac{1}{3}}$ and can be extended by λ on the complement of V . Without loss of generality, assume that τ_L is compactly supported in the interior of $V^{\frac{1}{4}}$. Theorem 1.2 implies

$$(4.16) \quad \kappa(V^{\frac{1}{4}}, \lambda_1, \tau_L^2) = \kappa(W, \lambda_1, \tau_L^2).$$

The Liouville domain $(V^{\frac{1}{4}}, \lambda_1)$ is isomorphic to the disk-cotangent bundle $D_\varepsilon^*\mathbb{S}^n$ for some radius ε because λ_1 is equal to λ_0 on $V^{\frac{1}{4}}$. Besides, the symplectomorphism τ_L^2 is symplectically isotopic to the fibered Dehn twist. Hence, due to Proposition 4.4,

$$(4.17) \quad \kappa(V^{\frac{1}{4}}, \lambda_1, \tau_L^2) = 4.$$

Finally, the 1-form $\lambda - \lambda_1$ is, by construction, the exterior derivative of a function $W \rightarrow \mathbb{R}$ that is equal to 0 near the boundary. Hence Lemma 2.6, (4.16), and (4.17) finish the proof. \square

Proof of Corollary 1.4. The corollary follows from Theorem 1.2 and Theorem 1.3. \square

4.3. An application to closed geodesics.

Definition 4.5. The **visible rank** $r(W, a)$ of a Liouville domain W and an admissible slope a is defined to be

$$r(W, a) := \dim_{\mathbb{Z}_2} \iota(HF(W, \text{id}, a)),$$

where

$$\iota : HF_*(W, \text{id}, a) \rightarrow HF_*(W, \text{id}, \infty) = SH_*(W; \mathbb{Z}_2)$$

is the natural morphism.

Lemma 4.6. *Let W_1, W_2 be as in Section 2.3. Assume there exists $r \in (0, 1)$ such that $W_2^r \subset W_1$. Then, the transfer morphism*

$$(4.18) \quad SH(W_2) = HF(W_2, \infty) \rightarrow HF(W_1, \infty) = SH(W_1)$$

is an isomorphism.

Proof. The map

$$(4.19) \quad \widehat{W}_1 \rightarrow \widehat{W}_2 : p \mapsto \begin{cases} j(p) & \text{for } p \in W_1 \\ (\text{pr}_1(j(p)), \text{pr}_2(j(x)) \cdot r) & \text{for } p = (x, r) \in \partial W_1 \times (0, \infty) \end{cases}$$

is a diffeomorphism that respects the Liouville forms. Here $j : W_1 \hookrightarrow W_2$ stands for the inclusion, and pr_1, pr_2 are the projections $\partial W_2 \times (0, \infty) \rightarrow \partial W_2$ and $\partial W_2 \times (0, \infty) \rightarrow (0, \infty)$, respectively. We will identify \widehat{W}_1 and \widehat{W}_2 via this map. By the assumption, there exists $s \in (0, \infty)$ such that $W_2 \subset W_1^s$. The transfer morphisms

$$\begin{aligned} SH_*(W_1^s) &\rightarrow SH_*(W_1), \\ SH_*(W_2) &\rightarrow SH_*(W_2^r) \end{aligned}$$

are isomorphisms due to Lemma 4.16 in [2]. They fit (because of the functoriality of the transfer morphisms [2]) into the commutative diagram

$$\begin{array}{ccccc} SH_*(W_1^s) & \longrightarrow & SH_*(W_2) & & \\ & \searrow & \downarrow & \searrow & \\ & & SH_*(W_1) & \longrightarrow & SH_*(W_2^r) \end{array}$$

Hence (4.18) is an isomorphism as well. \square

Theorem 4.7. *Let W_1, W_2 be as in Section 2.3. Assume there exists $r \in (0, 1)$ such that $W_2^r \subset W_1$. Then,*

$$r(W_2, a) \leq r(W_1, a)$$

for all slopes $a \in (0, \infty)$ that are admissible with respect to both W_1 and W_2 .

Proof. Since ∂W_2 is compact, the set of admissible slopes with respect to W_2 is open (see Lemma A.3). Let $\varepsilon > 0$ be such that the elements of $[a - \varepsilon, a]$ are all admissible with respect to W_2 . Lemma 2.25 in [7] implies

$$r(W_2, a) = r(W_2, a - \varepsilon).$$

Consider the commutative diagram

$$\begin{array}{ccc} SH_*(W_2) & \longrightarrow & SH_*(W_1) \\ \uparrow & & \uparrow \\ HF_*(W_2, a - \varepsilon) & \longrightarrow & H_*(W_1, a), \end{array}$$

whose existence follows from Theorem 1.1. The upper horizontal arrow of the diagram is an isomorphism due to Lemma 4.6. Hence

$$r(W_2, a) = r(W_2, a - \varepsilon) \leq r(W_1, a).$$

□

Corollary 4.8. *Let Q be a closed manifold, and let g_0 and g_1 be two Riemannian metrics on Q such that $g_1 \leq g_0$, i.e. $g_1(v, v) \leq g_0(v, v)$ for all $v \in TQ$. Assume*

$$(4.20) \quad r(D^*Q, a) > \dim_{\mathbb{Z}_2} H(Q, \mathbb{Z}_2),$$

*where D^*Q is the disk-cotangent bundle of Q with respect to g_0 and a is a positive real number that is not a period of some closed g_0 -geodesic on Q . Then there exists a closed g_1 -geodesic of length less than a .*

Proof. Denote D^*Q by W_2 and the disk-cotangent bundle of Q with respect to g_1 by W_1 . The condition $g_1 \leq g_0$ implies (see Lemma 4.10 below) $W_1 \subset W_2$.

We argue by contradiction, i.e. assume there exists no (non-constant) closed g_1 -geodesic on Q with the length less than a . Then, the numbers in $(0, a]$ are all admissible with respect to W_1 . Hence

$$r(W_1, a) = r(W_1, \varepsilon) \leq \dim HF(W_1, \varepsilon) = \dim H(W_1; \mathbb{Z}_2) = \dim H(Q; \mathbb{Z}_2),$$

where $\varepsilon > 0$ is small enough. This, together with (4.20), implies

$$r(W_1, a) < r(W_2, a),$$

a relation in contradiction with Theorem 4.7. □

Example 4.9. Let (\mathbb{S}^n, g_0) be the n -dimensional sphere with the standard Riemannian metric, and let $\varepsilon \in (0, 2\pi)$. The visible rank $r(D^*\mathbb{S}^n, 2\pi + \varepsilon)$ is equal to 6 and, therefore, greater than the sum of the Betty numbers of \mathbb{S}^n . Hence any Riemannian metric g on \mathbb{S}^n with $g \leq g_0$ has a non-constant closed geodesic of length less than or equal to $2\pi + \varepsilon$. Since ε is an arbitrary number in $(0, 2\pi)$, g has a non-constant closed geodesic of length less than or equal to 2π .

Lemma 4.10. *Let Q be a manifold, and let g_0, g_1 be two Riemann metrics on Q such that $g_1 \leq g_0$. Then,*

$$D^*Q(g_1) \subset D^*Q(g_0),$$

*where $D^*Q(g_i)$, $i = 0, 1$ stand for the disk cotangent bundle of Q with respect to g_i , $i = 0, 1$.*

Proof. It is enough to prove that, for every $\alpha \in T^*Q$, the inequality

$$|v|_0 \leq |w|_1$$

holds, where $|\cdot|_i$ is the norm induced by g_i , $i = 0, 1$, and v, w are unique vectors in TQ such that

$$g_0(v, \cdot) = \alpha = g_1(w, \cdot).$$

Using the polarization identity

$$2g_1(v, w) = |v|_1 + |w|_1 - |v - w|_1$$

and the inequality $|v|_0 \geq |v|_1$, we get

$$\begin{aligned} |w|_1^2 - |v|_0^2 &= |w|_1^2 + |v|_1^2 - |w - v|_1^2 - |v|_1^2 + |w - v|_1^2 - |v|_0^2 \\ &= 2g_1(w, v) - |v|_1^2 + |w - v|_1^2 - |v|_0^2 \\ &= 2\alpha(v) - |v|_1^2 + |w - v|_1^2 - |v|_0^2 \\ &= 2|v|_0^2 - |v|_1^2 + |w - v|_1^2 - |v|_0^2 \\ &= |v|_0^2 - |v|_1^2 + |w - v|_1^2 \geq 0. \end{aligned}$$

□

APPENDIX A. TECHNICAL LEMMAS

Lemma A.1. *Let r_0, β_0, β_1 be real numbers and let ℓ, α be positive real numbers. Then, the following conditions are equivalent.*

- *There exists a convex, strictly increasing function*

$$h : [r_0, r_0 + \ell] \rightarrow \mathbb{R}$$

such that the function

$$\mathbb{R} \rightarrow \mathbb{R} : r \mapsto \begin{cases} \beta_0 & r \leq r_0, \\ h(r) & r \in [r_0, r_0 + \ell], \\ \alpha r + \beta_1 & r \geq r_0 + \ell \end{cases}$$

is C^∞ .

- *The following inequalities hold*

$$\begin{aligned} (A.21) \quad & \alpha r_0 < \beta_0 - \beta_1, \\ & \beta_1 - \beta_0 + \alpha r_0 + \alpha \ell > 0. \end{aligned}$$

Proof. We first assume there exists such a function and prove (A.21). Since h is increasing, we get

$$\beta_0 = h(r_0) < h(r_0 + \ell) = \beta_1 + \alpha r_0 + \alpha \ell,$$

or equivalently

$$\beta_1 - \beta_0 + \alpha r_0 + \alpha \ell > 0.$$

The Newton-Leibniz formula, together with h being convex, implies

$$\alpha r_0 + \alpha \ell + \beta_1 - \beta_0 = h(r_0 + \ell) - h(r_0) = \int_{r_0}^{r_0 + \ell} h'(r) dr < \int_{r_0}^{r_0 + \ell} \alpha dr = \alpha \ell.$$

Hence (A.21). (The function h' is continuous and not constant. Therefore the strict inequality holds.)

Now, we construct h under assumption (A.21). Consider the famous cut-off function

$$\mathbb{R} \rightarrow \mathbb{R} : r \mapsto \begin{cases} e^{-\frac{1}{1-r^2}} & |r| < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ the function given by

$$f_2(r) := \frac{\alpha}{c\ell} f_1\left(\frac{2r - 2r_0 - \ell}{\ell}\right),$$

where

$$c := \int_{\mathbb{R}} f_1(r) dr.$$

It is bigger than 0 on $(r_0, r_0 + \ell)$ and equal to 0 elsewhere. In addition $\int_{\mathbb{R}} f_2(r) dr = \alpha$. Hence the function

$$f_3 : \mathbb{R} \rightarrow \mathbb{R} : r \mapsto \int_{r_0}^r f_2(t) dt$$

is equal to 0 on $(-\infty, r_0]$, strictly increasing on $(r_0, r_0 + \ell)$, and equal to α on $[r_0 + \ell, \infty)$.

If

$$\beta_1 - \beta_0 + r_0\alpha + \ell\alpha = \int_{r_0}^{r_0+\ell} f_3(r) dr =: c_1,$$

then the function

$$h(r) := \beta_0 + \int_{r_0}^r f_3(t) dt$$

satisfies the conditions. Assume $\beta_1 - \beta_0 + r_0\alpha + \ell\alpha > c_1$. Let $k_1 \geq 1$ be a real number such that

$$\int_{r_0}^{r_0+\ell} f_4(r) dr = \beta_1 - \beta_0 + r_0\alpha + \ell\alpha,$$

where

$$f_4(r) := f_3(k_1 r - k_1 r_0 + r_0).$$

Such a number exists because

$$g_1 : t \mapsto \int_{r_0}^{r_0+\ell} f_3(tr - tr_0 + r_0) dr$$

is a continuous function satisfying

$$\begin{aligned} g_1(1) &= c_1 < \beta_1 - \beta_0 + r_0\alpha + \ell\alpha, \\ \lim_{t \rightarrow \infty} g(t) &= \ell\alpha > \beta_1 - \beta_0 + r_0\alpha + \ell\alpha. \end{aligned}$$

The last inequality is due to (A.21). In this case, we can take h to be

$$h(r) := \beta_0 + \int_{r_0}^r f_4(t) dt.$$

Assume now $\beta_1 - \beta_0 + r_0\alpha + \ell\alpha < c_1$. Let $\varepsilon \in (0, 1]$ and $k_2 \in [1, \infty)$ be real numbers such that

$$\int_{r_0}^{r_0+\ell} f_3(r) f_5(r) dr = \beta_1 - \beta_0 + r_0\alpha + \ell\alpha,$$

where

$$f_5(r) := \varepsilon + \frac{1-\varepsilon}{\alpha} f_3(k_2 r - k_2 r_0 - k_2 \ell + r_0 + \ell).$$

The function

$$h(r) := \beta_0 + \int_{r_0}^{r_0+\ell} f_3(r) f_5(r) dr$$

satisfies the conditions. In what follows, we show that the numbers ε and k_2 exist. Assume the contrary. Since the function

$$g_2 : (0, 1] \times [1, \infty) : (s, t) \mapsto \int_{r_0}^{r_0+\ell} f_3(r) \left(s + \frac{1-s}{\alpha} f_3(tr - tr_0 - t\ell + r_0 + \ell) \right) dr$$

is continuous, $g_2(1, t) = c_1 > \beta_1 - \beta_0 + r_0\alpha + \ell\alpha$, and the space $(0, 1] \times [1, \infty)$ is connected, the inequality

$$g_2(s, t) > \beta_1 - \beta_0 + r_0\alpha + \ell\alpha$$

holds for all $(s, t) \in (0, 1] \times [1, \infty)$. The function $r \mapsto f_3(tr - tr_0 - t\ell + r_0 + \ell)$ is equal to 0 on $(-\infty, r_0 + \frac{t-1}{t}\ell)$, hence

$$g_2(s, t) \leq sc_1 + \frac{\ell}{t} \max f_3 \left(s + \frac{\max f_3}{\alpha} \right).$$

Therefore, for s small enough and t large enough, we get

$$g_2(s, t) < \beta_1 - \beta_0 + r_0\alpha + \ell\alpha$$

(the number on the right-hand side is positive due to (A.21)). This is a contradiction and the proof is finished. \square

Lemma A.2. *Let r_0, β_0, β_1 be real numbers and let ℓ, α be positive real numbers. Then, the following conditions are equivalent.*

- *There exists a concave, strictly increasing function*

$$h : [r_0, r_0 + \ell] \rightarrow \mathbb{R}$$

such that the function

$$\mathbb{R} \rightarrow \mathbb{R} : r \mapsto \begin{cases} \alpha r + \beta_0 & r \leq r_0, \\ h(r) & r \in [r_0, r_0 + \ell], \\ \beta_1 & r \geq r_0 + \ell \end{cases}$$

is C^∞ .

- *The following inequalities hold*

$$(A.22) \quad 0 < \beta_1 - \beta_0 - \alpha r_0 < \ell\alpha.$$

Moreover, given $\alpha_1 \in \mathbb{R}$, the inequality

$$h(r) > \alpha_1 r$$

holds for all $r \in [r_0, r_0 + \ell]$ if, and only if,

$$\alpha r_0 + \beta_0 > r_0 \alpha_1 \quad \text{and} \quad \beta_1 > \alpha_1 (r_0 + \ell).$$

Proof. The function $h : [r_0, r_0 + \ell] \rightarrow \mathbb{R}$ is concave, strictly increasing if, and only if, the function

$$[-r_0 - \ell, -r_0] \rightarrow \mathbb{R} : r \mapsto -h(-r)$$

is convex and strictly increasing. Hence Lemma A.1 implies the first part of the lemma. The second part follows from the concavity of the function h . \square

Lemma A.3. *Let M be a closed contact manifold with contact form α . Then, the set $S(M, \alpha)$ of all periods of periodic Reeb orbits on (M, α) is a closed subset of \mathbb{R} .*

Proof. Let $d : M \times M \rightarrow [0, \infty)$ be a metric on M . Denote by $\sigma_t : M \rightarrow M$ the Reeb flow of (M, α) . Consider the function

$$f : M \times \mathbb{R} \rightarrow [0, \infty) : (x, t) \mapsto d(x, \sigma_t(x)).$$

The set $f^{-1}(\{0\})$ is a closed subset of $M \times \mathbb{R}$, because f is a continuous map. Since M is compact, the projection

$$\text{pr}_2 : M \times \mathbb{R} \rightarrow \mathbb{R}$$

is a closed map. Hence

$$\text{pr}_2(f^{-1}(\{0\})) = S(M, \alpha)$$

is a closed subset of \mathbb{R} . □

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